MATH 54 - HINTS TO HOMEWORK 9

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Here are a couple of hints to Homework 9! Enjoy :)

NOTE: In case you're stuck, make sure to look at my 'Systems of Differential Equations'-Handouts. They should help you solve all the problems!

SECTION 9.4: LINEAR SYSTEMS IN NORMAL FORM

9.4.1, 9.4.2. Those problems are easier to do than to explain. For example, for $9.4.1$:

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}, \mathbf{f} = \begin{bmatrix} t^2 \\ e^t \end{bmatrix}$$

9.4.5. Here is the trick: Let $z(t) = y'(t)$. Then the equation $y''(t) - 3y'(t) - 10y(t) = \sin(t)$ becomes $z'(t) - 3z(t) - 10y(t) = \sin(t)$, so $z'(t) = 10y(t) + 3z(t) + \sin(t)$, and so we get the system:

$$\begin{cases} y'(t) = z(t) \\ z'(t) = 10y(t) + 3z(t) + \sin(t) \end{cases}$$

which you can (and should) read as:

$$\begin{cases} y'(t) = 0y(t) + z(t) + 0 \\ z'(t) = 10y(t) + 3z(t) + \sin(t) \end{cases}$$

So $A = \begin{bmatrix} 0 & 1 \\ 10 & 3 \end{bmatrix}$, and $\mathbf{f} = \begin{bmatrix} 0 \\ \sin(t) \end{bmatrix}$

9.4.13, 9.4.15. Use the Wronskian! The good news is that the wronskian is very easy to calculate! Just ignore any constants and put all the three vectors in a matrix. For example, for $9.4.15$, the (pre)-Wronskian is:

$$\widetilde{W}(t) = \begin{bmatrix} e^t & 5e^t \\ -3e^t & -15e^t \end{bmatrix}$$

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And as usual, pick your favorite point t_0, and evaluate $det(\widetilde{W}(t_0))$. If this is nonzero, your functions are linearly independent.

9.4.27. Fundamental matrix just means that each column is a solution to the system of DE and moreover the columns are linearly independent (so use the Wronskian). Then just use the formula $\mathbf{x}(t) = \mathbf{X}(t)\mathbf{X}^{-1}(t_0)\mathbf{x_0}$ with $t_0 = 0$ and $\mathbf{x}_0 = x(0)$. Notice that you only have to calculate the inverse of $\mathbf{X}$ at 0, **not** in general!

SECTION 9.5: HOMOGENEOUS LINEAR SYSTEMS WITH CONSTANT COEFFICIENTS

If you're lost about this, check out the handout 'Systems of differential equations' on my website!

9.5.11, 9.5.13, 9.5.14. Again, look at the handout on my website. All you have to do is to find the eigenvalues and eigenvectors of A and put them together.

Also, to deal with the 'finding the eigenvalues' part, remember the following theorem:

Rational roots theorem: If a polynomial p has a zero of the form $r = \frac{a}{b}$, then a divides the constant term of p and b divides the leading coefficient of p.

This helps you 'guess' a zero of p. Then use long division to factor out p.

Finally, in case you find a repeated root, just treat it as a simple root! (i.e don't put stuff like te^t)

9.5.17. First, draw two lines, one spanned by $\mathbf{u_1}$ and the other one spanned by $\mathbf{u_2}$. Then on the first line, draw arrows pointing *away* from the origin (because of the e^{2t}-term in the solution, points on that line *move away* from the origin). On the second line, draw arrows pointing *towards* the origin (because of the e^{-2t}-term, solutions move towards the origin). Finally, for all the other points, all you have to do is to 'connect' the arrows (think of it like drawing a force field or a velocity field).

If you want a picture of how the answer looks like, google 'saddle phase portrait differential equations' and under images, check out the second image you get!

9.5.19, 9.5.21. The fundamental matrix is just the matrix whose columns are the solutions to your differential equation. Basically find the general solution to your differential equation, ignore the constants, and put everything else in a matrix!

SECTION 9.6: COMPLEX EIGENVALUES

9.6.1, 9.6.3, 9.6.7. Again, same thing as usual, find the eigenvalues and eigenvectors of A. However, you only need to do half of the work, just as in section 5.5 from the linear algebra book! Use the formula at the bottom of page 598.

9.6.13. If you're running out of time, here's a nice trick: See the formula in 9.4.26 on page 584.

SECTION 9.7: NONHOMOGENEOUS LINEAR EQUATIONS

Again, the handout 'Systems of differential equations' goes through this in more detail!

Note: In what follows, $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ are $2-$vectors.

9.7.1. Guess $\mathbf{f}(t) = \mathbf{a}$

9.7.5. Guess $\mathbf{f}(t) = e^{-2t}(\mathbf{a}t + \mathbf{b})$

9.7.11, 9.7.13, 9.7.15. The formula is:

$$\left(\widetilde{W}(t)\right) \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \mathbf{f}$$

where $\widetilde{W}(t)$ is the (pre)-Wronskian, or fundamental matrix for your system (essentially the solutions but without the constants).

Note: If you're running out of time, use formula (11) on page 605.

SECTION 9.8: THE MATRIX EXPONENTIAL FUNCTIONS

9.8.1, 9.8.5. Here's the formula: If $(A - rI)^k = \mathbf{0}$ for some k, then:

$$e^{At} = e^{rt}\left(I + (A - rI)t + (A - rI)^2 \frac{t^2}{2} + \cdots + (A - rI)^{k-1} \frac{t^{k-1}}{(k-1)!}\right)$$

So for 9.8.1, $k = 2$ and $\lambda = 3$, so:

$$e^{At} = e^{3t}\left(I + (A - 2I)t\right)$$

And for 9.8.5, $k = 3$ and $\lambda = -2$, so:

$$e^{At} = e^{-2t}\left(I + (A + 2I)t + (A + 2I)^2 \frac{t^2}{2}\right)$$

9.8.7. Use $e^{At} = Pe^{Dt}P^{-1}$, where D is your matrix of eigenvalues, and P is your matrix of eigenvectors.